

Greediness and Equilibrium in Congestion Games

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Abstract

Rosenthal (1973) introduced the class of congestion games and proved that they always possess a Nash equilibrium in pure strategies. Fotakis et al. (2005) introduce the notion of a greedy strategy tuple, where players sequentially and irrevocably choose a strategy that is a best response to the choice of strategies by former players. Whereas the former solution concept is driven by strong assumptions on the rationality of the players and the common knowledge thereof, the latter assumes very little rationality on the players' behavior. From Fotakis [4] it follows that for Tree Representable congestion Games greedy behavior leads to a NE. In this paper we obtain necessary and sufficient conditions for the equivalence of these two solution concepts. Such equivalence enhances the viability of these concepts as realistic outcomes of the environment. The conditions for such equivalence to emerge for monotone symmetric games is that the strategy set has a tree-form, or equivalently is a 'extension-parallel graph'.

1 Introduction

Congestion games form a natural class of games that are useful in modeling many realistic settings, such as traffic and communication networks, routing, load balancing and more.

A symmetric congestion game is a 4-tuple $(N, R, \Sigma, \{P_r\}_{r \in R})$, where N is a finite set of players, R is a finite set of resources, $\Sigma \subset 2^R$ is the set of players' strategies, and

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for any $r \in R$, $P_r : N \rightarrow \mathbb{R}$ is the resource's payoff function. A strategy of a player is a choice of a subset resources, $s \in \Sigma$. For any strategy tuple $s = (s^i)_{i \in N} \in \Sigma^N$ let $c(s)_r = |\{i \in N : r \in s^i\}|$ denote the number of players that utilize r (a.k.a. the congestion of the resource r) and denote by $c(s) = (c(s)_1, \dots, c(s)_r)$ the congestion vector. The utility of a player is the total payment for the resources she utilizes. Formally, $U^i(s) = \sum_{r \in s^i} P_r(c(s)_r)$.

A congestion game is *monotone* if for any $1 \leq k < l \leq N$ and $r \in R$, $P_r(k) > P_r(l)$. Monotone congestion games widely prevail in modeling traffic and communication problems, production resource allocation and more. In *single-signed* congestion games payments are either all positive or all negative. Typically, whenever monotone congestion games are used for modeling, they are assumed single-signed.

A *congestion game form* is a pair $F = \{R, \Sigma\}$, composed of the set of resources and a set of strategies (subsets of R). For any congestion game $G = (N, R, \Sigma, \{P_r\}_{r \in R})$ let $F(G) = (R, \Sigma)$ denote the corresponding game form. Given a congestion game form F , let $\mathcal{G}(F) = \{G : (F(G) = F) \wedge (G \text{ monotone})\}$ denote the class of all monotone congestion games with the game form F .

We say that a strategy set, $\Sigma \subset R$, is *subset-free* if for any $s \neq t \in \Sigma$ we have $s \not\subset t$ and $s \not\supset t$. Thus, a *subset-free Congestion Game (Form)* is a Congestion Game (Form) with a subset-free strategy space. For any equilibrium analysis of single-signed monotone congestion games the assumption of subset free strategy sets is without loss of generality. In particular, note that in such games for any pair of strategies $s \subset t$ in Σ either s is dominated by t (in case resource payments are all positive) or t is dominated by s (in case resource payments are all negative) and so after deletion of dominated strategies we are left with subset free sets.

As usual, a profile $s \in \Sigma^N$ is a pure NE of G , if for each player i , for each strategy $t^i \in \Sigma$, $U^i(s^i, s^{-i}) \geq U^i(t^i, s^{-i})$, where s^{-i} is the vector of strategies of all players but i . Informally, a set of strategies is a Nash equilibrium if no player can do better by unilaterally deviating. The set of all pure NE of a congestion game G will be denoted $NE(G)$.

1.1 Known Results

Congestion games were introduced by Rosenthal (1973) [10], who proved that any congestion game has a Nash equilibrium in pure strategies. In spite of this fact there is still valid concern about the prevalence of a Nash equilibrium in reality. There are two classical criticisms over the validity of a Nash equilibrium profile as a *solution concept* which can be made - one that is computationally driven and another that is rationality driven.

The latter criticism is based on the fact that for an equilibrium to prevail players must have common knowledge of rationality, a condition typically unrealistic. The former criticism argues that the existence of a pure Nash equilibrium does not imply it is computationally simple to find such an equilibrium. In particular, whenever the strategy space is rich this may be a challenging endeavor. In general, this may require searching over all strategy tuples, whose number can grow exponentially with the number of players. However, as congestion games have the *finite improvement property* one could suspect that it may be easier to find such an equilibrium.¹ However, it turns out that improvement paths can be exponentially long, as demonstrated by Ieong et al. [7] and Fabricant et al. [2]. Recently Fabrikant et al. [2] provide an algorithm that finds a Nash equilibrium in polynomial time, for an important subset of congestion games. This is done via a reduction to a flow problem, yet leaves little insight regarding the nature of the NE.

Fotakis et al. (2005) [3] introduce the notion of a *greedy strategy profile*. Let us consider a dynamic setting with the players joining the game sequentially. Each player, upon arrival, irrevocably chooses a best response strategy to the choice of strategies of the previous players, while ignoring subsequent players. The resulting strategy profile is called a *greedy strategy profile*. Let us denote by $Z(G)$ the set of all greedy strategy profiles. Note the two degrees of freedom in the process - the order of the agents and the tie-breaking rule in case of indifference among several options.

Formally, $s \in Z(G)$, if there exists a permutation $\pi : N \rightarrow N$ of the players ($\pi(i)$ denotes the order of i) such that for any player i who chooses strategy s^i we have $\sum_{r \in s^i} P_r(c(s_\pi^i)_r + 1) \geq \sum_{r \in t} P_r(c(s_\pi^i)_r + 1) \forall t \in \Sigma$, where $c(s_\pi^i)_r = |\{j : r \in s^{\pi(j)}, \pi(j) < \pi(i)\}|$ is the number players preceding i according to the permutation π whose strategy includes resource r . Clearly $Z(G) \neq \emptyset$, and typically $Z(G)$ may contain many such profiles.

In contrast with the rationality assumption underlying the notion of Nash equilibrium, the rationality requirement from a greedy profile is very low, possibly too low, as players clearly choose to ignore anything they do not observe. In addition, calculating a greedy equilibrium profile is a much less demanding task than calculating a Nash equilibrium. Hypothetically, whenever $NE(G) = Z(G)$ the prevalence of an equilibrium is much more likely. This is because the identity of the two sets suggests that the rationality assumption underlying an equilibrium profile is weak and the complexity of finding such an equilibrium is typically linear with respect to the number of players. This motivates us to study the relationship between the sets $NE(G)$ and $Z(G)$.

Fotakis et al. [3] have already shown that $Z(G) \subset NE(G)$ for simple congestion games. Fotakis [4] showed that if a class of congestion games that satisfy two conditions: (1)

¹The finite improvement property asserts that if players sequentially improve their utility by unilateral strategy changes then this process is finite and must end in a Nash equilibrium profile, see, Monderer and Shapley [9].

the game form is that of ‘extension-parallel graph’, namely one can map the resources to the set of edges in a extension-parallel graph and the strategies are the set of paths leading from a certain node in the graph (designated as the source node) to another node (designated as the target node); and (2) the resource payoff functions satisfy a property referred to as the ‘Common Best Reply’ requirement, met in symmetric congestion game. In particular, their result implies that $Z(G) \subset NE(G)$ for simple congestion games, where strategies are the singleton resource subsets of R .

Additional papers that study conditions under which $Z(G) \subset NE(G)$ are Ackerman et al. [1] and Fotakis [4]. In Ackerman et al. [1] the main observation is that greedy best responses converge very fast to a NE, when the strategy structure is that of a Matroid, while in Fotakis [4] shows a theorem from which follows that in Tree Representable Congestion Games greedy leads to NE.

1.2 Our Contribution

This paper characterizes the setting for which $Z(G)$ and $NE(G)$ coincide. In particular, our main result argues that a necessary and sufficient conditions for these two solution concepts to coincide is that the the game form is that of ‘extension-parallel graph’. These results extend the state of the art knowledge in two ways. First, it is shown that for such game forms not only is every greedy profile a Nash equilibrium but also vice versa. In addition, we show that for such equivalence to hold for a given game form it must be the case that the game form is of a certain class, namely a ‘extension-parallel graph’. In particular, given a game form not satisfying this condition, we show how to construct resource payoff functions such that the the set of NE profiles and greedy profiles will not coincide.

The ‘extension-parallel graph’ game form is also the necessary and sufficient condition for the set of NE profiles to coincide with the set of strong equilibrium profiles, as shown by Holzman and Law Yone [5] and [6]. Note that Holzman and Law Yone [6] refer ‘tree representable’ game form which, a-priori, are different than ‘extension-parallel graph’ game forms, but they go on and prove equivalence (Theorem 1 and Theorem 2).

Combining the results in [5] and [6] and our contribution, we obtain equivalence for ‘extension-parallel graph’ game forms (or Tree Representable game forms) between greedy profiles and strong NE. Moreover, if the game is not form is not ‘extension parallel graph’ it is possible to find payoffs where the equivalence will not hold.

The structure of the article is as follows: Section 2 provides a variety of examples that demonstrate that without any restrictions on the game form there is no connection between the sets $NE(G)$ and $Z(G)$. Section 3 formalizes the notion of tree representable

games, and discusses the characteristics of this class. Then in section 4 we present and prove the main result, namely equivalence between $NE(G)$ and $Z(G)$ for tree representable congestion games.

2 Examples

Here we provide several examples for the various relations between $Z(G)$ and $NE(G)$. As we shall demonstrate those can differ depending on the game in question.

Example 2.1 $NE(G) \cap Z(G) = \emptyset$ - Greedy profiles and equilibria are mutually exclusive.

In this example there are 3 players and 3 resources. The strategy space is the set of all pairs of resources.

# of players / Resource	A	B	C
1	10	10	8
2	8	4	6
3	1	1	5

The unique greedy profile (up to renaming of players) is (AB, AC, BC) . Note this is not a Nash equilibrium since player 1 can profitably deviate from AB to AC , increasing her utility from $8+4$ to $8+5$. On the other hand the unique Nash equilibrium (up to renaming of players) is AC, AC, BC , and obtained after this deviation.

Example 2.2 $Z(G) \subsetneq NE(G)$ - Greedy profiles strictly contained in NE profiles.

In this example there are 2 players and 4 resource. Each strategy must contain one of the resources A, B and one of the resources C, D .

# of players / Resource	A	B	C	D
1	40	30	20	15
2	10	11	12	13

Clearly the unique greedy profile is (AC, BD) , which is also a Nash equilibrium. However, there is an additional Nash equilibrium profile (AD, BC) .

Example 2.3 $NE(G) \subsetneq Z(G)$ - The greedy profiles strictly contain the NE.

In this example there are 2 players and 3 resources. The strategy space is the set of all pairs of resources.

# of players / Resource	A	B	C
1	10	8	8
2	1	7	6

The greedy profiles are AB, BC and AC, BC . The first one is the unique pure NE of the game.

Example 2.4 $Z(G) \cap NE(G) \neq \emptyset$, $Z(G) \setminus NE(G) \neq \emptyset$ and $NE(G) \setminus Z(G) \neq \emptyset$.

Consider a game with 2 players, 5 resources: A, B, C, D, E and the strategy space $\Sigma = \{AB, AC, DB, E\}$.

# of players / Resource	A	B	C	D	E
1	-1	-1	-5	-2	-10
2	-5	-10	-100	-100	-100

If we express this game in the standard bi-matrix form we get:

Game	AB	AC	DB	E
AB	-15, -15	-6, -10	-11, -12	-2, -10
AC	-10, -6	-105, -105	-6, -3	-6, -10
DB	-12, -11	-3, -6	-110, -110	-3, -10
E	-10, -2	-10, -6	-10, -3	-100, -100

It is now easy to verify that the only two Nash equilibria of this game are (AB, E) and (AC, DB) . The greedy profiles, on the other hand, are (AB, E) and (AB, AC) .

Example 2.5 $NE(G) = Z(G)$ - Equivalence

This holds in any simple congestion game (follows from our main result).

3 Tree Representable Congestion Games

Holzman and Law Yone (1996) [5] study a class of game forms called *Tree Representable Congestion Games* (TRCG). To define this they introduce the notion of an *R-tree*. An *R-tree* is a tree whose nodes, except for the root, are labeled by elements in R , each appearing at most once (however not necessarily all elements in R are mapped to the nodes). With each terminal node in the tree we can associate the set of resources that form the unique path leading from the root to that node. Thus, the *R-tree* induces a set of strategies.

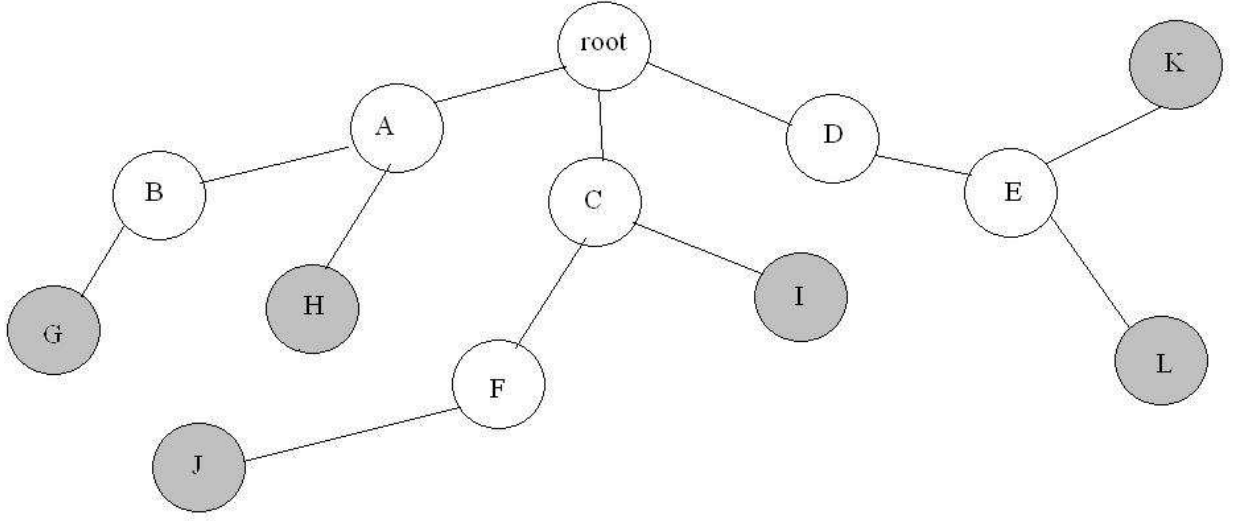


Figure 1: Example of an R-Tree

The game form (R, Σ) is *Tree Representable* if there exists an R -tree which induces the set Σ . A congestion game G is tree representable iff its corresponding game form is tree representable.²

Example 3.1 Consider a game form (R, Σ) with $R = \{A, B, C, D, E, F, G, H, I, J, K, L\}$ and $\Sigma = \{ABG; AH; CI; CFJ; DEK; DEL\}$. This game form is tree representable and the corresponding tree is depicted in Figure 1, where the set of terminal nodes is $\{G, H, I, J, K, L\}$. Note that the tree in the figure is not unique. For example, we can exchange nodes F and J and still represent the same game.

3.1 When is a game form Tree Representable?

A *Bad Configuration* is a combination of 2 resources A, B and 3 strategies s_1, s_2, s_3 such that the following three conditions are satisfied:

1. $A, B \in s_1$
2. $A \in s_2 \setminus s_3$

² Actually, Holzman and Law Yone (1996) [5] allow for strategies that are induced by paths that do not necessarily lead to terminal nodes. However, whenever the game is subset-free this cannot occur and hence our focus is only on terminal nodes. In [6] they introduce the notion of subset free and adjust the tree representable definition.

3. $B \in s_3 \setminus s_2$

Note that in subset-free congestion games there must be an additional resource $z \in s_2 \setminus s_1$, as s_2 is not a subset of s_1 . Similarly, there exists a resource $w \in s_3 \setminus s_1$ (possibly $w = z$). Verifying whether a game form is tree representable is possible in polynomial time due to the following result of Holzman and Law Yone (1996) [5]:

Theorem 3.1 *A congestion game form does not have a bad configuration if and only if it is tree representable.*

Thus, verifying whether a game form is tree representable can be done by going over all possible pairs of resources and triplets of strategies, which is polynomial in both factors. Note that if the game is tree representable, each strategy must have at least one unique resource (the last resource on the tree path), implying that the number of resources is equal to or larger than the number of strategies.

4 Main Result

Our main result links tree representable congestion games with the equivalence of the two solution concepts based on greediness and equilibrium:

Theorem 4.1 *Let F be a subset free Congestion Game Form. F is tree representable iff for any congestion game $G \in \mathcal{G}(F)$, we have that $Z(G) = NE(G)$.*

Recall that the set of strategies that survives deletion of dominated strategies in single-signed monotone congestion games is subset-free. Thus, the following conclusion immediately follows from Theorem 4.1:

Corollary 4.1 *Let F be a Congestion Game Form. F is tree representable iff for any congestion game $G \in \hat{\mathcal{G}}(F)$, we have that $Z(G) = NE(G)$,*

where $\hat{\mathcal{G}}(F)$ denotes the set of single signed monotone congestion games with the game form F .

We split the proof of Theorem 4.1 into two propositions, one showing that tree representability is sufficient for the desired equivalence and the other showing it is necessary.

4.1 Tree Representability is Sufficient

As noted in [6] TRCG are equivalent to extension parallel network games. For such games Fotakis [4], showed in Theorem 1 the following:

Theorem 4.2 *For any n player symmetric congestion game on an extension parallel network, every best response sequence reaches a pure NE in at most n steps.*

From here it is easy to show the following lemma:

Lemma 4.1 *Let G be a TRCG. Then $Z(G) \subseteq NE(G)$.*

Proof:

Let L be the lowest resource payoff which can be attained in G . Let us add a resource r_l to the game G with the resource payoffs $L - 1, L - 2, \dots, L - N$. Let us add to the set of strategies of G the strategy r_l . Let us denote the extended game as G' . Let s be a strategy profile of G' where all players select the resource r_l .

Let z be a greedy behavior strategy profile of G with the ordering π and a draw breaking rule τ .

Let us select the first player according to the ordering π and denote her i_1 . Let us relocate her to the best responses to s^{-i_1} , in case of several the strategy selected by τ . Let us denote the obtained profile as s_1 . Similarly, let us select the second agent in π and relocate her to the best response to $s_1^{-i_2}$, in case of several the strategy selected by τ . After continuing in such manner once for every agent we will obtain the strategy profile s^N . Following theorem 4.2, s^N is a NE of G' .

Note that the best response of any agent is a strategy in G , as the payoff from staying on r_l is strictly lower than relocating to any strategy in G , no matter the congestion on all other resources. Therefore, this mechanism is identical to greedy behavior of the original game, and since after N steps no player selects r_l we obtain a strategy profile of G . Note that any NE of G' is also a NE of G , as G' has one strategy more than G , and besides the two games are identical. Therefore, s^N is a NE of G .

Therefore, any greedy strategy profile with ordering π and draw breaking rule τ is a NE of the game G .

□

Lemma 4.2 *If F is tree representable then in any $G \in \mathcal{G}(F)$, $NE(G) \subset Z(G)$.*

Proof:

The claim is trivial for any single player game. Let us assume it holds for $n - 1$ and show that it must also hold for n players. Let $s = (s^1, \dots, s^n) \in NE(G)$. We will assume, without loss of generality, that utility of agent n is the lowest among all agents:

$$U^n(s) \leq U^i(s) \quad \forall i. \quad (1)$$

We denote by $P(s)$ the projection of the strategy tuple s onto players $1, \dots, n - 1$. We argue that $P(s)$ is a Nash equilibrium of the game with $n - 1$ agents.

Assume it is not, then there exists an agent j who can profitably deviate to some strategy σ . Let $\tilde{P}(s) = (P(s)^{-j}, \sigma)$ denote the strategy tuple of the $n - 1$ players after such deviation. Formally, $\tilde{s} = (s^{-j}, \sigma)$, and we can say that:

$$U^j(P(\tilde{s})) > U^j(P(s)). \quad (2)$$

Let us denote by $\bar{s} = (s^{-n}, \sigma)$, the strategy profile obtained from s by replacing the strategy of agent n with σ . Similarly, denote by \check{s} the strategy tuple obtained from s by replacing agent j 's strategy with s^n , namely: $\check{s} = (s^{-j}, s^n)$. As s is a Nash equilibrium:

$$U^n(s) \geq U^n(\bar{s}), \quad U^j(s) \geq U^j(\check{s}). \quad (3)$$

Note the following connection between the corresponding congestion vectors:

- $\forall r \in s^j \cap s^n, C(s)_r - 1 = C(P(\check{s}))_r = C(P(s))_r;$
- $\forall r \in s^j \setminus s^n, C(s)_r = C(P(s))_r;$ and
- $\forall r \in s^n \setminus s^j, C(P(\check{s}))_r = C(s)_r.$

Therefore:

$$\begin{aligned} U^j(P(s)) &= \sum_{r \in s_j \cap s_n} P_r(C(s)_r - 1) + \sum_{r \in s_j \setminus s_n} P_r(C(s)_r) \\ U^j(P(\check{s})) &= \sum_{r \in s_j \cap s_n} P_r(C(s)_r - 1) + \sum_{r \in s_n \setminus s_j} P_r(C(s)_r) \end{aligned}$$

which implies

$$U^j(P(\check{s})) - U^j(P(s)) = \sum_{r \in s_n \setminus s_j} P_r(C(s)_r) - \sum_{r \in s_j \setminus s_n} P_r(C(s)_r). \quad (4)$$

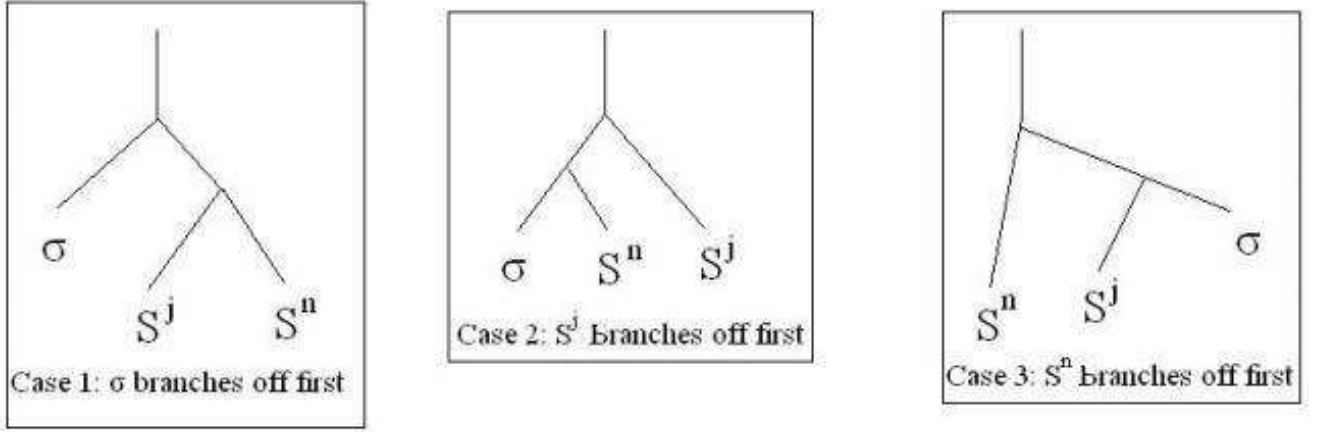


Figure 2: The three cases as described by the lemma

Since we know that $U^n(s) \leq U^j(s)$, the difference denoted in equation 4 is non negative. Thus, we can say that:

$$U^j(P(\tilde{s})) \leq U^j(P(s)). \quad (5)$$

Note that the three strategies, s^j, s^n, σ are not all identical, as s^j and σ are different.

The game G is tree representable and therefore the following 3 cases, depicted in figure 2, exhaust all the possibilities on the connection among the three strategies s^j, s^n, σ which differ on the strategy that branches off the tree path first (they are not necessarily mutually exclusive):

- Case 1 - $\sigma \cap s^j \cap s^n = \sigma \cap s^j = \sigma \cap s^n$ (the path representing σ branches off first).
- Case 2 - $\sigma \cap s^j \cap s^n = s^j \cap \sigma = s^j \cap s^n$ (the path representing s^j branches off first).
- Case 3 - $\sigma \cap s^j \cap s^n = s^n \cap \sigma = s^n \cap s^j$ (the path representing s^n branches off first).

Case 1 - $\sigma \cap s^j \cap s^n = \sigma \cap s^j = \sigma \cap s^n$.

Since s is a NE, we have that $U^j(s) \geq U^j(\tilde{s})$. Therefore:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(s)_r) + \sum_{r \in s^j \cap s^n \cap \sigma} P_r(C(s)_r) - \sum_{r \in s^j \cap s^n \cap \sigma} P_r(C(\tilde{s})_r) - \sum_{r \in \sigma \setminus s^j} P_r(C(\tilde{s})_r) \geq 0.$$

Note that $\forall r \in s^j \cap s^n \cap \sigma, C(s)_r = C(\tilde{s})_r$. Therefore, the second and third element cancel out. In addition, $\forall r \in s^j \setminus \sigma, C(P(s))_r \leq C(s)_r$ and $\forall r \in \sigma \setminus s^j, C(P(\tilde{s}))_r = C(\tilde{s})_r$. As P_r are strictly decreasing, the last inequality implies:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(P(s))_r) - \sum_{r \in \sigma \setminus s^j} P_r(C(P(\tilde{s}))_r) \geq 0.$$

Moreover, as $C(P(s))_r = C(P(\tilde{s}))_r \quad \forall r \in s^j \cap s^n \cap \sigma$:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(P(s))_r) + \sum_{r \in s^j \cap \sigma} P_r(C(P(s))_r) - \sum_{r \in \sigma \setminus s^j} P_r(C(P(\tilde{s}))_r) - \sum_{r \in s^j \cap \sigma} P_r(C(P(\tilde{s}))_r) \geq 0.$$

The last inequality implies that $U^j(P(s)) \geq U^j(P(\tilde{s}))$ which contradicts inequality 2.

Case 2 - $\sigma \cap s^j \cap s^n = s^j \cap \sigma = s^j \cap s^n$.

We know that s is a NE, thus $U^n(s) \geq U^n(\bar{s})$. Thus:

$$\sum_{r \in s^n \setminus \sigma} P_r(C(s)_r) + \sum_{r \in \sigma \cap s^n} P_r(C(s)_r) \geq \sum_{r \in \sigma \setminus s^n} P_r(C(\bar{s})_r) - \sum_{r \in \sigma \cap s^n} P_r(C(\bar{s})_r)$$

Note that $\forall r \in s^n \cap \sigma, C(\bar{s})_r = C(s)_r$. Therefore the second and fourth element cancel out:

$$\sum_{r \in s^n \setminus \sigma} P_r(C(s)_r) - \sum_{r \in \sigma \setminus s^n} P_r(C(\bar{s})_r) \geq 0.$$

Additionally, $\forall r \in \sigma \setminus s^n, C(\bar{s})_r = C(P(\tilde{s}))_r$. Similarly, $\forall r \in s^n \setminus \sigma, C(s)_r = C(P(\tilde{s}))_r$. Thus:

$$\sum_{r \in s^n \setminus \sigma} P_r(C(P(\tilde{s}))_r) - \sum_{r \in \sigma \setminus s^n} P_r(C(P(\tilde{s}))_r) \geq 0.$$

Moreover, $\forall r \in s^n \cap \sigma, C(P(\tilde{s}))_r = C(P(\tilde{s}))_r$. Thus:

$$\sum_{r \in s^n \setminus \sigma} P_r(C(P(\tilde{s}))_r) + \sum_{r \in s^n \cap \sigma} P_r(C(P(\tilde{s}))_r) - \sum_{r \in \sigma \setminus s^n} P_r(C(P(\tilde{s}))_r) - \sum_{r \in s^n \cap \sigma} P_r(C(P(\tilde{s}))_r) \geq 0.$$

We get that $U^j(P(\tilde{s})) \geq U^j(P(\tilde{s}))$. Combined with inequality 5 we reach a contradiction with inequality 2.

Case 3 - $\sigma \cap s^j \cap s^n = s^n \cap \sigma = s^n \cap s^j$.

As s is a NE, we know that $U^j(s) \geq U^j(\tilde{s})$. Therefore:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(s)_r) + \sum_{r \in (s^j \cap \sigma)} P_r(C(s)_r) \geq \sum_{r \in \sigma \setminus s^j} P_r(C(\tilde{s})_r) + \sum_{r \in (s^j \cap \sigma)} P_r(C(\tilde{s})_r).$$

Note that $\forall r \in s^j \cap \sigma, C(\tilde{s})_r = C(s)_r$, thus the second and fourth element cancel out. Additionally, $\forall r \in s^j \cap \sigma, C(P(\tilde{s}))_r = C(P(s))_r$. Therefore:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(s)_r) + \sum_{r \in s^j \cap \sigma} P_r(C(P(s))_r) \geq \sum_{r \in \sigma \setminus s^j} P_r(C(\tilde{s})_r) + \sum_{r \in s^j \cap \sigma} P_r(C(P(\tilde{s}))_r).$$

Moreover, $\forall r \in (s^j \setminus \sigma)$, $C(P(s))_r = C(s)_r$ and $\forall r \in (\sigma \setminus s^j)$, $C(\tilde{s})_r = C(P(\tilde{s}))_r$. Therefore:

$$\sum_{r \in s^j \setminus \sigma} P_r(C(P(s))_r) + \sum_{r \in s^j \cap \sigma} P_r(C(P(s))_r) \geq \sum_{r \in \sigma \setminus s^j} P_r(C(P(\tilde{s}))_r) + \sum_{r \in s^j \cap \sigma} P_r(C(P(\tilde{s}))_r).$$

The last inequality states that $U^j(P(s)) \geq U^j(P(\tilde{s}))$, once again contradicting inequality 2.

Thus, $P(s)$ must be a Nash equilibrium for the $n - 1$ players. Using the induction hypothesis we conclude that $P(s)$ is a greedy profile for players $1, \dots, n - 1$. As player n best-responds to $P(s)$ (recall that s is a Nash equilibrium) we conclude that s is a greedy profile, as desired. \square

Combining lemmas 4.1 and 4.2 yields the following direction in the statement of Theorem 4.1:

Proposition 4.1 *Let F be a subset free congestion game form. If F is tree representable then $G \in \mathcal{G}(F) \implies Z(G) = NE(G)$.*

4.2 Tree Representability is Necessary

We show that if a game form is non tree representable then it can be coupled with monotone resource payoff function to yields a game, G , without the equivalence of $NE(G)$ and $Z(G)$. Let $F = (R, \Sigma)$ be a game form that is not tree representable. By Theorem 3.1 there must exist 2 resources, $A, C \in R$ and three strategies, $s_1, s_2, s_3 \in \Sigma$ such that $A \in (s^1 \cap s^3) \setminus s^2$ and $C \in (s^1 \cap s^2) \setminus s^3$.

Recall that F is assumed subset free. In particular there must exist resources B and D that satisfy $B \in s^2 \setminus s^1$ and $D \in s^3 \setminus s^1$. We now argue that $B \neq D$ and more broadly that:

Lemma 4.3 *If for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then $(s^2 \cap s^3) \setminus s^1 = \emptyset$.*

Proof:

Suppose that there exists some resource $E \in (s^2 \cap s^3) \setminus s^1$. Then we have that:

$$s^1 \cap \{A, C, E\} = \{A, C\}$$

$$s^2 \cap \{A, C, E\} = \{C, E\}$$

$$s^3 \cap \{A, C, E\} = \{A, E\}$$

Consider a 2 player game with the following payoff functions:

# of players / Resource	A	C	E	other resources
1	10	9	8	$\frac{1}{M}$
2	1	6	7	$\frac{1}{2M}$

Let M be sufficiently large to ensure that $\frac{2|R|}{M} < 1$. This game does not satisfy $Z(G) = NE(G)$: for example, the NE involving AE and EC cannot be attained in a greedy manner. Thus we reach a contradiction. \square

Corollary 4.2 *B and D are two different resources.*

Lemma 4.4 *If for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then there exists a strategy $s^4 \neq s^2$ such that $s^4 \subset s^2 \cup (s^3 \setminus s^1)$.*

Proof:

By Lemma 4.3 $(s^2 \cap s^3) \setminus s^1 = \emptyset$. Therefore the following assignment of resource payment functions for 2 players determines a monotone congestion game (where M is an arbitrary large number satisfying $M > R^9$):

Resource Set	$P_r(1)$	$P_r(2)$
$s^1 \cap s^2 \cap s^3$	$-1/M^2$	$-1/R$
$(s^1 \cap s^2) \setminus s^3$	$-1/M^2$	$- R ^6$
$(s^1 \cap s^3) \setminus s^2$	$-1/ R ^5$	$-2M$
$s^1 \setminus (s^2 \cup s^3)$	$-1/ R ^5$	$-2M$
$s^2 \setminus (s^1 \cup s^3)$	$-1/ R $	$-2M$
$s^3 \setminus (s^2 \cup s^1)$	$-1/ R ^4$	$-2M$
$(s^1 \cup s^2 \cup s^3)^c$	$-M$	$-2M$

Note that resources not in s^1 are worse than any resource in s^1 by a factor of $|R|$. Therefore the first greedy player must select the strategy s^1 . The second greedy agent will need to select resources in $s^2 \cup (s^3 \setminus s^1)$, as all other resources have utility of at least $-M$. Assume the statement of the lemma is incorrect and that the only such strategy is s^2 and so player 2 chose strategy s^2 . Now note that the first greedy agent has a profitable deviation from s^1 to s^3 , avoiding the high negative payoff on resources in $(s^1 \cap s^2) \setminus s^3$ (e.g., on resource C). This implies that $Z(G) \neq NE(G)$ and a contradiction is reached. \square

Combined with subset-freeness of the strategy set we can now conclude:

Corollary 4.3 *There exists a resource in s^4 which is also in $s^3 \setminus (s^1 \cup s^2)$. We can assume WLOG that this resource is D.*

Proposition 4.2 *Let F be a subset free congestion game form. Then $Z(G) = NE(G) \forall G \in \mathcal{G}(F)$ implies F is tree representable.*

Proof:

Suppose F is not tree representable and $Z(G) = NE(G) \forall G \in \mathcal{G}(F)$. Then, by Lemmas 4.3 and 4.4, there exist 4 strategies s^1, \dots, s^4 and 4 distinct resources, A, B, C, D satisfying: $A \in (s^1 \cap s^3) \setminus s^2$, $B \in s^2 \setminus s^1$, $C \in (s^1 \cap s^2) \setminus s^3$ and $D \in s^3 \setminus (s^1 \cup s^2)$. We will now show that in such case $s^4 \subset s^3$, thus contradicting the subset-freeness assumption.

Suppose there exists a resource $E \in s^4 \setminus s^3$. This in turn implies that $E \in s^2 \setminus s^3$. Let us distinguish between the case where $E \in s^1$ and the case $E \notin s^1$.

Case 1 - Assume $E \in s^1$. This, in particular implies that:

$$s^1 \cap \{A, D, E\} = \{A, E\}$$

$$s^3 \cap \{A, D, E\} = \{A, D\}$$

$$s^4 \cap \{A, D, E\} = \{D, E\}$$

Consider a 2 player game with the following payoff functions:

# of players / Resource	A	D	E	other resources
1	10	9	8	$\frac{1}{M}$
2	1	6	7	$\frac{1}{2M}$

Let M be sufficiently large to ensure that $\frac{2|R|}{M} < 1$. This game does not satisfy $Z(G) = NE(G)$, as for example the NE involving AE and DE cannot be attained by greedy behavior. Thus we reach a contradiction.

Case 2 - Assume $E \notin s^1$. This, in particular implies that:

$$s^1 \cap \{A, C, D, E\} = \{A, C\}$$

$$s^2 \cap \{A, C, D, E\} = \{E, C\}$$

$$s^3 \cap \{A, C, D, E\} = \{A, D\}$$

$$s^4 \cap \{A, C, D, E\} = \{E, D\}$$

Consider the 2 player game with the following resource payment functions:

# of players / Resource	A	E	C	D	other resources
1	40	30	20	15	$\frac{1}{M}$
2	10	11	12	13	$\frac{1}{2M}$

Let M be sufficiently large to ensure that $\frac{2|R|}{M} < 1$. This game is in the spirit Example 2.2 (with the addition of extra resources that yield negligible utility). Similar to Example 2.2 this game does not satisfy $Z(G) = NE(G)$, thus reaching a contradiction. \square

The proof of Theorem 4.1 follows from Propositions 4.1 and 4.2.

5 Summary

Monotone congestion games are a well proved modeling tool. In many realistic cases one can easily assume that the strategy set is subset-free (e.g., when strategies are paths leading from a source node to a target node) or that resource payment functions are single-signed (e.g., they express latency over a graph edge and are hence negative). We show that in such cases the set of pure Nash equilibria coincides with the set of greedy strategy profiles. We conclude that in such cases a Nash equilibrium forms a viable solution concept as it emerges from very weak rationality assumptions and does not hinge on common knowledge of rationality. In addition, it can be the case that the computational difficulty of finding the equilibria set in such games is substantially weaker than in an arbitrary game or even an arbitrary congestion game.

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